

PREDUALS FOR SPACES OF OPERATORS INVOLVING HILBERT SPACES AND TRACE-CLASS OPERATORS

HANNES THIEL

ABSTRACT. Continuing the study of preduals of spaces $\mathcal{L}(H, Y)$ of bounded, linear maps, we consider the situation that H is a Hilbert space. We establish a natural correspondence between isometric preduals of $\mathcal{L}(H, Y)$ and isometric preduals of Y .

The main ingredient is a Tomiyama-type result which shows that every contractive projection that complements $\mathcal{L}(H, Y)$ in its bidual is automatically a right $\mathcal{L}(H)$ -module map.

As an application, we show that isometric preduals of $\mathcal{L}(S_1)$, the algebra of operators on the space of trace-class operators, correspond to isometric preduals of S_1 itself (and there is an abundance of them). On the other hand, the compact operators are the unique predual of S_1 making its multiplication separately weak* continuous.

1. INTRODUCTION

A predual of a Banach space X is a Banach space F together with an isomorphism $X \cong F^*$. For conceptual reasons, it is more useful to consider preduals of X as subsets of X^* . More precisely, a closed subspace $F \subseteq X^*$ is an (isometric) predual of X if for the inclusion map $\iota_F: F \rightarrow X^*$, the transpose map $\iota_F^*: X^{**} \rightarrow F^*$ restricts to an (isometric) isomorphism $X \xrightarrow{\cong} F^*$.

Every predual of X induces a corresponding weak* topology $\sigma(X, F)$. Two preduals of X are equal (as subsets of X^*) if and only if they induce the same weak* topology. Due to the importance of weak* topologies in functional analysis, it is an interesting problem to study the existence and uniqueness of preduals of Banach spaces. We refer to the survey of Godefroy, [God89], and the references therein.

The space X is said to have a *strongly unique isometric predual* if there exists an isometric predual $F \subseteq X^*$ and if $F = G$ for every isometric predual $G \subseteq X^*$. Of course, every reflexive space X has a strongly unique isometric predual, namely X^* . By Sakai's theorem, every von Neumann algebra has a strongly unique isometric predual. This applies in particular to the von Neumann algebra $\mathcal{L}(H)$ of bounded, linear maps on a Hilbert space H . The isometric predual of $\mathcal{L}(H)$ is $H \hat{\otimes} H$, the projective tensor product of H and H .

This can be generalized to other spaces of operators. Given Banach spaces X and Y , we let $\mathcal{L}(X, Y)$ denote the space of operators (that is, bounded linear maps) from X to Y . There is a canonical isometric isomorphism $\mathcal{L}(X, F^*) \cong (X \hat{\otimes} F)^*$. It follows that every isometric predual of Y induces an isometric predual of $\mathcal{L}(X, Y)$. The question is whether every isometric predual of $\mathcal{L}(X, Y)$ occurs in this way.

Date: 6 March 2017.

2010 Mathematics Subject Classification. Primary: 47L05, 47L10, Secondary: 47L45, 46B10, 46B20.

Key words and phrases. Preduals, unique predual, dual Banach algebra, dual module, complemented subspace.

Godefroy and Saphar showed that this is the case for many classes of Banach spaces. More precisely, they proved that $X \hat{\otimes} F$ is the strongly unique isometric predual of $\mathcal{L}(X, F^*)$ if X and F satisfy the Radon Nikodým property (RNP); see [GS88, Proposition 5.10]. Thus, if X satisfies the RNP, and if Y has an isometric predual satisfying the RNP, then $\mathcal{L}(X, Y)$ has a strongly unique isometric predual.

The main result of this paper extends this to the case that X is a Hilbert space H and Y is arbitrary: We show that every isometric predual of $\mathcal{L}(H, Y)$ is induced by an isometric predual of Y ; see Theorem 2.7. In particular, $\mathcal{L}(H, Y)$ has a (strongly unique) isometric predual if and only if Y does; see Corollary 2.8.

The strategy to obtain these results is as follows: First, there is a natural correspondence between (isometric) preduals of Y and (isometric) preduals of $\mathcal{L}(X, Y)$ that make the right action by $\mathcal{L}(X)$ weak* continuous. In fact, this holds for arbitrary Banach spaces X and Y , as shown in [GT16, Theorem 6.7]. Thus, in order to obtain the desired correspondence between isometric preduals of Y and isometric preduals of $\mathcal{L}(X, Y)$, we consider the condition that every isometric predual of $\mathcal{L}(X, Y)$ automatically makes the right action by $\mathcal{L}(X)$ weak* continuous. This does not hold for arbitrary X and Y ; see Remark 2.14. However, we show that this second condition is verified whenever X is a Hilbert space or the space of trace-class operators on a Hilbert space, and Y is arbitrary.

To obtain the result about automatic weak* continuity of the right $\mathcal{L}(X)$ -action, we use that an isometric predual $F \subseteq \mathcal{L}(X, Y)^*$ corresponds to a contractive projection $\pi_F: \mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ with weak* closed kernel. The condition that F makes the right action by $\mathcal{L}(X)$ weak* continuous corresponds to the condition that π_F is a right $\mathcal{L}(X)$ -module map. Hence, we are naturally faced with the following problem:

Problem 1.1. Find conditions on X and Y that guarantee that every contractive projection $\pi_F: \mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ is automatically a right $\mathcal{L}(X)$ -module map.

It was shown by Tomiyama that every contractive projection from a C^* -algebra A onto a sub- C^* -algebra B is automatically a conditional expectation, which means exactly that it is a B -bimodule map. We therefore consider a positive solution to Problem 1.1 a Tomiyama-type result. Adapting the proof of Tomiyama's result, we obtain a positive solution to Problem 1.1 whenever X is a Hilbert space; Theorem 2.4.

In Section 3, we show that our results hold when the Hilbert space H is replaced with the space of trace-class operators $\mathcal{S}_1(H)$. Given a Banach space X , we consider the condition that Problem 1.1 has a positive solution for every Y . We show that this condition passes to projective tensor products; see Lemma 3.2. Then the results for the trace-class operators follow using that $\mathcal{S}_1(H)$ is isometrically isomorphic with the projective tensor product $H \hat{\otimes} H$.

It follows that isometric preduals of $\mathcal{L}(\mathcal{S}_1(H))$ naturally correspond to isometric preduals of $\mathcal{S}_1(H)$; see Example 3.9. Since the diagonal operators in $\mathcal{S}_1(H)$ form a weak* closed isometric copy of ℓ^1 , and since ℓ^1 has many different isometric preduals, it follows that $\mathcal{S}_1(H)$ has many different isometric preduals as well. On the other hand, we show that the ‘standard’ predual of compact operators is the unique predual of $\mathcal{S}_1(H)$ making its multiplication separately weak* continuous; see Theorem 3.10.

ACKNOWLEDGEMENTS

The author would like to thank Eusebio Gardella and Tim de Laat for valuable feedback. The author was partially supported by the Deutsche Forschungsgemeinschaft (SFB 878 *Groups, Geometry & Actions*).

NOTATION

Given Banach spaces X and Y , an *operator* from X to Y is a bounded, linear map $X \rightarrow Y$. We let $\mathcal{L}(X, Y)$ denote the space of operators $X \rightarrow Y$. We use $X \hat{\otimes} Y$ to denote the projective tensor product of X and Y .

We let $\kappa_X: X \rightarrow X^{**}$ denote the canonical isometric map from X to its bidual. Using κ_X , we identify X with a subspace of X^{**} .

2. PREDUALS INVOLVING HILBERT SPACES

2.1. Given Banach spaces X and Y , the space $\mathcal{L}(X, Y)$ has a natural $\mathcal{L}(Y)$ - $\mathcal{L}(X)$ -bimodule structure. Given $a \in \mathcal{L}(X)$, the action of a is given by $R_a: \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Y)$, $R_a(f) := f \circ a$, for $f \in \mathcal{L}(X, Y)$. Thus, a acts by precomposing on the right of $\mathcal{L}(X, Y)$. Similarly, the action of $b \in \mathcal{L}(Y)$ is given by postcomposing on the left, that is, by $L_b: \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Y)$, $L_b(f) := b \circ f$, for $f \in \mathcal{L}(X, Y)$.

These actions induce a $\mathcal{L}(X)$ - $\mathcal{L}(Y)$ -bimodule structure on $\mathcal{L}(X, Y)^*$. The *left* action of $a \in \mathcal{L}(X)$ on $\mathcal{L}(X, Y)^*$ is given by R_a^* . The *right* action of $b \in \mathcal{L}(Y)$ on $\mathcal{L}(X, Y)^*$ is given by L_b^* . Similarly, we obtain a $\mathcal{L}(Y)$ - $\mathcal{L}(X)$ -bimodule structure on $\mathcal{L}(X, Y)^{**}$, with $a \in \mathcal{L}(X)$ acting on the right of $\mathcal{L}(X, Y)^{**}$ by R_a^{**} , and analogous for the left action of $\mathcal{L}(Y)$ on $\mathcal{L}(X, Y)^{**}$.

Given a C^* -algebra A and operators $a, b, x, y \in A$ with $a^*b = 0$, we have $\|ax + by\|^2 \leq \|ax\|^2 + \|by\|^2$, which is an analog of Bessel's inequality; see [Bla06, II.3.1.12, p.66]. We first prove two versions of this result in a more general context.

Lemma 2.2. *Let H be a Hilbert space, let X be a Banach space, let $a, b \in \mathcal{L}(H)$ satisfy $a^*b = 0$, and let $f, g \in \mathcal{L}(X, H)$. Then*

$$\|af + bg\|^2 \leq \|af\|^2 + \|bg\|^2.$$

Proof. The equation $a^*b = 0$ implies that the range of a and b are orthogonal. Given $x \in X$, it follows that the elements afx and bgx are orthogonal in H , whence

$$\|afx + bgx\|^2 = \|afx\|^2 + \|bgx\|^2.$$

Using this at the second step, we deduce that

$$\|af + bg\|^2 = \sup_{\|x\| \leq 1} \|afx + bgx\|^2 = \sup_{\|x\| \leq 1} (\|afx\|^2 + \|bgx\|^2) \leq \|af\|^2 + \|bg\|^2,$$

as desired. \square

Lemma 2.3. *Let H be a Hilbert space, let Y be a Banach space, and let $a, b \in \mathcal{L}(H)$ satisfy $ab^* = 0$. Given $f, g \in \mathcal{L}(H, Y)$, we have*

$$\|fa + gb\|^2 \leq \|fa\|^2 + \|gb\|^2.$$

*Similarly, given $F, G \in \mathcal{L}(H, Y)^{**}$, we have $\|Fa + Gb\|^2 \leq \|Fa\|^2 + \|Gb\|^2$.*

Proof. Let us show the first inequality. We identify H^* with \overline{H} , the complex conjugate of H . The transpose of the operator $a: H \rightarrow H$ is an operator $\overline{H} \rightarrow \overline{H}$, which we denote by \overline{a} . Similarly, we consider $\overline{b} \in \mathcal{L}(\overline{H})$. It follows from $ab^* = 0$ that $\overline{a}^*\overline{b} = 0$.

Consider the transposed operators f^* and $(fa)^*$ in $\mathcal{L}(Y^*, H^*) = \mathcal{L}(Y^*, \overline{H})$. Then $(fa)^* = \overline{a}f^*$, where $\overline{a}f^*$ is given by the left action of $\mathcal{L}(\overline{H})$ on $\mathcal{L}(Y^*, \overline{H})$. Similarly, we have $(gb)^* = \overline{b}g^*$ and $(fa + gb)^* = \overline{a}f^* + \overline{b}g^*$. Applying Lemma 2.2 at the third step, we compute

$$\|fa + gb\|^2 = \|(fa + gb)^*\|^2 = \|\overline{a}f^* + \overline{b}g^*\|^2 \leq \|\overline{a}f^*\|^2 + \|\overline{b}g^*\|^2 = \|fa\|^2 + \|gb\|^2,$$

as desired.

Let us show the second inequality. Let p and q be the right support projections of a and b in $\mathcal{L}(H)$, respectively. Then $ap = a$, $bq = b$, and $pq^* = 0$. It follows that $Fa = Fap$ and $Gb = Gbq$.

Using Goldstine's theorem, we choose nets $(f_i)_i$ and $(g_j)_j$ in $\mathcal{L}(H, Y)$ such that $(f_i)_i$ converges weak* to Fa , such that $(g_j)_j$ converges weak* to Gb , and such that $\|f_i\| \leq \|Fa\|$ for all i and $\|g_j\| \leq \|Gb\|$ for all j . Then $(f_i p)_i$ converges weak* to Fap . Using this at the second step, we deduce that

$$\|Fa\| = \|Fap\| \leq \varliminf_i \|f_i p\| \leq \overline{\lim}_i \|f_i p\| \leq \|Fa\|,$$

and hence $\lim_i \|f_i p\| = \|Fa\|$. Analogously, we obtain that $\lim_j \|g_j q\| = \|Gb\|$.

Using this at the third step, using that the net $(f_i p + g_j q)_{i,j}$ converges weak* to $Fa + Gb$ at the first step, and using the first inequality of this lemma at the second step, we deduce that

$$\|Fa + Gb\|^2 \leq \varliminf_{i,j} \|f_i p + g_j q\|^2 \leq \varliminf_{i,j} (\|f_i p\|^2 + \|g_j q\|^2) = \|Fa\|^2 + \|Gb\|^2,$$

as desired. \square

Let A be a C^* -algebra and let $B \subseteq A$ be a sub- C^* -algebra. By Tomiyama's theorem, every contractive operator $\pi: A \rightarrow B$ satisfying $\pi(b) = b$ for every $b \in B$ is automatically a left and right B -module map (called a conditional expectation), that is, we have $\pi(ba) = b\pi(a)$ and $\pi(ab) = \pi(a)b$ for all $a \in A$ and $b \in B$. The next result is in the same spirit and with a similar proof as Tomiyama's theorem. It provides a partial positive solution to Problem 1.1.

Theorem 2.4. *Let H be a Hilbert space, let Y be a Banach space. Then every contractive projection $\pi: \mathcal{L}(H, Y)^{**} \rightarrow \mathcal{L}(H, Y)$ is automatically a right $\mathcal{L}(H)$ -module map, that is, we have $\pi(Fa) = \pi(F)a$ for every $F \in \mathcal{L}(H, Y)^{**}$ and $a \in \mathcal{L}(H)$.*

Proof. First, we show the result for the case that a is a projection. Let $p \in \mathcal{L}(H)$ satisfy $p = p^2 = p^*$, and set $q := 1 - p$. The following argument is an adaption of the proof of Tomiyama's theorem as presented in [Bla06, Theorem II.6.10.2, p.132]. Let $\lambda > 0$. We have $\pi(\pi(Fp)q)q = \pi(Fp)q$. Using this at the first step, using that $\|\pi\| \leq 1$ and $\|q\| \leq 1$ at the third step, and using Lemma 2.3 at the fourth step, we deduce

$$\begin{aligned} (1 + \lambda)^2 \|\pi(Fp)q\|^2 &= \|\pi(Fp)q + \lambda\pi(\pi(Fp)q)q\|^2 \\ &= \|\pi(Fp + \lambda\pi(Fp)q)q\|^2 \\ &\leq \|Fp + \lambda\pi(Fp)q\|^2 \\ &\leq \|Fp\|^2 + \|\lambda\pi(Fp)q\|^2 = \|Fp\|^2 + \lambda^2 \|\pi(Fp)q\|^2. \end{aligned}$$

It follows that

$$(1 + 2\lambda) \|\pi(Fp)q\|^2 \leq \|Fp\|^2.$$

Since this holds for every $\lambda > 0$, we deduce that $\pi(Fp)q = 0$. Adding $\pi(Fp)p$ to this equation, we obtain

$$\pi(Fp) = \pi(Fp)p.$$

Switching the place of p and q in the above argument, we get $\pi(Fq)p = 0$. Adding $\pi(Fp)p$, we obtain that $\pi(F)p = \pi(Fp)p$ and combined with the above we deduce that

$$\pi(Fp) = \pi(Fp)p = \pi(F)p,$$

as desired.

Next, we use that every operator on a Hilbert space is a linear combination of finitely many projections. In fact, it was first shown by Fillmore that every operator on a separable Hilbert space is a linear combination of at most 257 projections.

This was improved by Pearcy and Topping, who removed the separability assumption and who also showed that 16 projections suffice; see [PT67, Corollary 2.3]. (Later, Matsumoto showed that even 10 projections suffice; see the introduction of [Mar10].) \square

2.5. Let X and Y be Banach spaces. Given $x \in X$, we let $\text{ev}_x: \mathcal{L}(X, Y) \rightarrow Y$ denote the evaluation map, given by $\text{ev}_x(f) := f(x)$, for $f \in \mathcal{L}(X, Y)$. In [GT16, Definition 4.7], we introduced a natural map $\alpha_{X,Y}: \mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y^{**})$ that satisfies

$$\alpha_{X,Y}(F)(x) = \text{ev}_x^{**}(F),$$

for $F \in \mathcal{L}(X, Y)^{**}$ and $x \in X$; see also [GT16, Lemma 4.19]. The map $\alpha_{X,Y}$ is always a right $\mathcal{L}(X)$ -module map.

Recall that $\kappa_Y: Y \rightarrow Y^{**}$ denotes the canonical isometric embedding of Y into its bidual. Let $\pi: Y^{**} \rightarrow Y$ be a projection, that is, $\pi \circ \kappa_Y = \text{id}_Y$. Define $\pi_*: \mathcal{L}(X, Y^{**}) \rightarrow \mathcal{L}(X, Y)$ by $\pi_*(f)(x) := \pi(f(x))$, for $f \in \mathcal{L}(X, Y^{**})$ and $x \in X$. Set $r_\pi := \pi_* \circ \alpha_{X,Y}: \mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$. Note that

$$r_\pi(F)x = \pi(\text{ev}_x^{**}(F)),$$

for $F \in \mathcal{L}(X, Y)^{**}$ and $x \in X$. The map r_π was considered in [GT16, Section 5], where it was also shown that r_π is a contractive projection. Moreover, r_π is a right $\mathcal{L}(X)$ -module map, that is, we have $r_\pi(Fa) = r_\pi(F)a$ for every $F \in \mathcal{L}(X, Y)^{**}$ and $a \in \mathcal{L}(X)$.

Recall that (concrete) preduals of Y are in natural bijection with projections $Y^{**} \rightarrow Y$ that have weak* closed kernel; see for example [GT16, Proposition 3.10]. Every predual induces a weak* topology. A predual of $\mathcal{L}(X, Y)$ makes the right action by each element from $\mathcal{L}(X)$ weak* continuous if and only if the associated projection $\mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ is a right $\mathcal{L}(X)$ -module map; see for example [GT16, Proposition B.8].

Theorem 2.6 ([GT16, Theorem 5.7]). *Let X and Y be Banach spaces. Given a projection $\pi: Y^{**} \rightarrow Y$, let $r_\pi: \mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ be given as in Paragraph 2.5. Then r_π is a projection that is a right $\mathcal{L}(X)$ -module map. Assigning to π the map r_π is a natural bijection between the following classes.*

- (1) *projections $Y^{**} \rightarrow Y$;*
- (2) *projections $\mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ that are right $\mathcal{L}(X)$ -module maps.*

Further, we have $\|\pi\| = \|r_\pi\|$. Moreover, the kernel of π is weak closed if and only if the kernel of r_π is. Thus, the above correspondence restricts to a natural bijection between (isometric) preduals of Y and (isometric) preduals of $\mathcal{L}(X, Y)$ that make the right action by $\mathcal{L}(X)$ weak* continuous.*

For the case that X is a Hilbert space, we may apply Theorem 2.4 to deduce that every contractive projection $\mathcal{L}(X, Y)^{**} \rightarrow \mathcal{L}(X, Y)$ is automatically a right $\mathcal{L}(X)$ -module map. In combination with Theorem 2.6, we obtain the main result of this paper:

Theorem 2.7. *Let H be a Hilbert space, and let Y be a Banach space. Assigning to a projection $\pi: Y^{**} \rightarrow Y$ the projection $r_\pi: \mathcal{L}(H, Y)^{**} \rightarrow \mathcal{L}(H, Y)$, as in Theorem 2.6, establishes a natural bijection between the following classes.*

- (1) *contractive projections $Y^{**} \rightarrow Y$;*
- (2) *contractive projections $\mathcal{L}(H, Y)^{**} \rightarrow \mathcal{L}(H, Y)$.*

Restricted to projections with weak closed kernel, we obtain that the above correspondence gives a natural bijection between isometric preduals of Y and isometric preduals of $\mathcal{L}(H, Y)$.*

Corollary 2.8. *Let H be a Hilbert space, and let Y be a Banach space. Then $\mathcal{L}(H, Y)$ is 1-complemented in its bidual if and only if Y is. Further, $\mathcal{L}(H, Y)$ has an isometric predual if and only if Y does. Moreover, if $\mathcal{L}(H, Y)$ has a strongly unique isometric predual, then so does Y , and vice versa.*

Corollary 2.9. *Let H be a Hilbert space, let Y be a Banach space, and let $F \subseteq \mathcal{L}(H, Y)^*$ be an isometric predual. Then for each $a \in \mathcal{L}(H)$, the right action of a on $\mathcal{L}(H, Y)$ is continuous for the weak* topology induced by F .*

Remark 2.10. Let X and Y be Banach spaces. There is a canonical isometric isomorphism

$$(X \hat{\otimes} Y)^* \cong \mathcal{L}(X, Y^*).$$

Given an isometric predual $F \subseteq Y^*$, with inclusion map $\iota_F: F \rightarrow Y^*$, the restriction of $\iota_F^*: Y^{**} \rightarrow F^*$ to Y is an isometric isomorphism. We obtain isometric isomorphisms

$$\mathcal{L}(X, Y) \cong \mathcal{L}(X, F^*) \cong (X \hat{\otimes} F)^*.$$

Theorem 2.7 states that if H is a Hilbert space then every isometric predual of $\mathcal{L}(H, Y)$ occurs this way. In particular, given an isometric isomorphism $\mathcal{L}(H, Y) \cong G^*$ for some Banach space G , there is an isometric isomorphism $H \hat{\otimes} F \cong G$, for some isometric predual $F \subseteq Y^*$.

Remark 2.11. It was shown by Godefroy and Saphar, [GS88, Proposition 5.10], that $X \hat{\otimes} Y$ is the strongly unique isometric predual of $\mathcal{L}(X, Y^*)$ if X and Y satisfy the Radon Nikodým property (RNP). Every reflexive space (in particular, every Hilbert space) satisfies the RNP. Thus, if Y has an isometric predual satisfying the RNP, then Corollary 2.8 follows from the result of Godefroy and Saphar. However, not every Banach spaces with strongly unique isometric predual occurs as the dual of a space satisfying the RNP; see Example 2.12. Thus, Theorem 2.7 and Corollary 2.8 go beyond the result in [GS88].

Example 2.12. Let M be a von Neumann algebra. By Sakai's theorem, M has a strongly unique isometric predual, denoted by M_* . It follows from Theorem 2.7 that $H \hat{\otimes} M_*$ is the strongly unique isometric predual of $\mathcal{L}(H, M)$. By [Chu81, Theorem 4], M_* satisfies the RNP if and only if M is a direct sum of type I factors.

Let \mathcal{R} denote the hyperfinite II_1 -factor. Then \mathcal{R}_* does not have the RNP, yet $H \hat{\otimes} \mathcal{R}_*$ is the strongly unique isometric predual of $\mathcal{L}(H, \mathcal{R})$.

Question 2.13. Does Theorem 2.7 hold when the Hilbert space is replaced by a general Banach space satisfying the RNP? More modestly, if X is an L^p -space, do isometric preduals of $\mathcal{L}(X, Y)$ correspond to isometric preduals of Y ?

Remark 2.14. Note that every positive solution of Problem 1.1 leads to an analog of Theorem 2.7. Therefore, Question 2.13 has a positive answer if the following instance of Problem 1.1 has a positive solution: Given a measure space μ and a Banach space Y , is every contractive projection $\mathcal{L}(L^p(\mu), Y)^{**} \rightarrow \mathcal{L}(L^p(\mu), Y)$ automatically a right $\mathcal{L}(L^p(\mu))$ -module map?

Consider the space ℓ^∞ of bounded sequences. Since ℓ^∞ is a von Neumann algebra, it has a strongly unique isometric predual. Thus, if Problem 1.1 had a positive solution for $X = Y = \ell^\infty$, then $\mathcal{L}(\ell^\infty)$ would have a strongly unique isometric predual. However, it was noted in [GS88, Remark 5.12] that $\mathcal{L}(\ell^\infty)$ has nonisomorphic isometric preduals. It follows in particular that there exists a contractive projection $\mathcal{L}(\ell^\infty)^{**} \rightarrow \mathcal{L}(\ell^\infty)$ that is not a right $\mathcal{L}(\ell^\infty)$ -module map.

3. PREDUALS INVOLVING TRACE-CLASS OPERATORS

Throughout this section H denotes a Hilbert space. We let $\mathcal{K}(H)$ and $\mathcal{S}_1(H)$ denote the compact and trace-class operators on H , respectively.

An operator $a \in \mathcal{L}(H)$ belongs to $\mathcal{S}_1(H)$ if and only if for some (equivalently, every) orthonormal basis $(e_j)_j$ of H the sum $\sum_j \langle |a|e_j, e_j \rangle$ is finite. Given $a \in \mathcal{S}_1(H)$ and an orthonormal basis $(e_j)_j$ of H , the sum $\sum_j \langle ae_j, e_j \rangle$ converges absolutely. Moreover, this value is independent of the choice of a orthonormal basis and we call

$$\mathrm{tr}(a) := \sum_j \langle ae_j, e_j \rangle$$

the *trace* of a . We set $\|a\|_1 := \mathrm{tr}(|a|)$. This defines a norm on $\mathcal{S}_1(H)$, turning the trace-class operators into a Banach space. Note that $\mathcal{S}_1(H)$ is a (non-closed) two-sided ideal in $\mathcal{L}(H)$. Moreover, we have $\mathcal{S}_1(H) \subseteq \mathcal{K}(H)$.

3.1. Given $a \in \mathcal{L}(H)$, the map $\mathcal{S}_1(H) \rightarrow \mathbb{C}$, $x \mapsto \mathrm{tr}(ax)$, is a bounded, linear functional on $\mathcal{S}_1(H)$. This induces an isometric isomorphism $\mathcal{L}(H) \cong \mathcal{S}_1(H)^*$, and we often identify the dual of $\mathcal{S}_1(H)$ with $\mathcal{L}(H)$. It is also well known that $\mathcal{K}(H) \subseteq \mathcal{L}(H) = \mathcal{S}_1(H)^*$ is an isometric predual of $\mathcal{S}_1(H)$. Thus, we have isometric isomorphisms

$$\mathcal{K}(H)^* \cong \mathcal{S}_1(H), \quad \text{and} \quad \mathcal{S}_1(H)^* \cong \mathcal{L}(H).$$

Lemma 3.2. *Let us say that a Banach space X has property $(*)$ if the following holds: For every Banach space E , every contractive projection $\mathcal{L}(X, E)^{**} \rightarrow \mathcal{L}(X, E)$ is a right $\mathcal{L}(X)$ -module map.*

Then, if two Banach spaces X and Y satisfy $()$, then so does their projective tensor product $X \hat{\otimes} Y$.*

Proof. Assume that X and Y satisfy $(*)$. Let E be another Banach space, and let $\pi: \mathcal{L}(X \hat{\otimes} Y, E)^{**} \rightarrow \mathcal{L}(X \hat{\otimes} Y, E)$ be a contractive projection.

Banach spaces form a closed monoidal category for the projective tensor product and with $\mathcal{L}(Y, _)$ adjoint to $_ \hat{\otimes} Y$. That is, there is a natural isometric isomorphism

$$\mathcal{L}(X \hat{\otimes} Y, E) \cong \mathcal{L}(X, \mathcal{L}(Y, E)).$$

An operator $f: X \hat{\otimes} Y \rightarrow E$ is identified with the operator $\tilde{f}: X \rightarrow \mathcal{L}(Y, E)$ that sends $x \in X$ to the operator $\tilde{f}(x): Y \rightarrow E$ given $\tilde{f}(x)(y) := f(x \otimes y)$, for $y \in Y$.

The projection π corresponds to a contractive projection $\pi': \mathcal{L}(X, \mathcal{L}(Y, E))^{**} \rightarrow \mathcal{L}(X, \mathcal{L}(Y, E))$. Let $\alpha_{X, \mathcal{L}(Y, E)}: \mathcal{L}(X, \mathcal{L}(Y, E))^{**} \rightarrow \mathcal{L}(X, \mathcal{L}(Y, E)^{**})$ be given as in Paragraph 2.5. Applying that X satisfies $(*)$ to the projection π' , there exists a unique contractive projection $\tau: \mathcal{L}(Y, E)^{**} \rightarrow \mathcal{L}(Y, E)$ such that $\pi' = \tau_* \circ \alpha_{X, \mathcal{L}(Y, E)}$. The situation is shown in the following diagram:

$$\begin{array}{ccc} \mathcal{L}(X \hat{\otimes} Y, E)^{**} & \cong & \mathcal{L}(X, \mathcal{L}(Y, E))^{**} \xrightarrow{\alpha_{X, \mathcal{L}(Y, E)}} \mathcal{L}(X, \mathcal{L}(Y, E)^{**}) \\ \uparrow \Bigg\downarrow \pi & & \uparrow \Bigg\downarrow \pi' \swarrow \tau_* \\ \mathcal{L}(X \hat{\otimes} Y, E) & \cong & \mathcal{L}(X, \mathcal{L}(Y, E)) \end{array}.$$

Let $F \in \mathcal{L}(X \hat{\otimes} Y, E)^{**}$, which we identify with $\tilde{F} \in \mathcal{L}(X, \mathcal{L}(Y, E))^{**}$ as explained above. Then

$$\pi(F)(x \otimes y) = [\pi'(\tilde{F})(x)](y) = \tau(\mathrm{ev}_x^{**}(\tilde{F}))(y),$$

for $x \in X$ and $y \in Y$.

Applying that Y satisfies $(*)$ to the projection τ , there exists a unique contractive projection $\sigma: Y^{**} \rightarrow Y$ such that

$$\tau(G)(y) = \sigma(\mathrm{ev}_y^{**}(G)),$$

for every $G \in \mathcal{L}(Y, E)^{**}$ and $y \in Y$. We claim that σ is the desired projection to verify that $X \hat{\otimes} Y$ satisfies (*).

Given $f \in \mathcal{L}(X \hat{\otimes} Y, E)$ with corresponding element $\tilde{f} \in \mathcal{L}(X, \mathcal{L}(Y, E))$, note that $\text{ev}_{x \otimes y}(f) = \text{ev}_y(\text{ev}_x(\tilde{f}))$. It follows that $\text{ev}_{x \otimes y}^{**}(F) = \text{ev}_y^{**}(\text{ev}_x^{**}(\tilde{F}))$. Using this at the last step, we deduce that

$$\pi(F)(x \otimes y) = \tau(\text{ev}_x^{**}(\tilde{F}))(y) = \sigma(\text{ev}_y^{**}(\text{ev}_x^{**}(\tilde{F}))) = \sigma(\text{ev}_{x \otimes y}^{**}(F)),$$

for every $x \in X$ and $y \in Y$. Thus, we have $\pi(F)(t) = \sigma(\text{ev}_t^{**}(F))$, for every simple tensor t in $X \hat{\otimes} Y$. It follows from linearity and continuity of the involved maps that the same equation holds for every $t \in X \hat{\otimes} Y$, as desired. \square

Remark 3.3. Theorem 2.7 states exactly that Hilbert spaces satisfy the condition (*) considered in Lemma 3.2.

Lemma 3.4. *Let \mathcal{S}_1 be the trace-class operators on some Hilbert space H , and let Y be a Banach space. Then every contractive projection $\mathcal{L}(\mathcal{S}_1, Y)^{**} \rightarrow \mathcal{L}(\mathcal{S}_1, Y)$ is automatically a right $\mathcal{L}(\mathcal{S}_1)$ -module map.*

Proof. It is well known that the trace-class operators on H are isometrically isomorphic to the projective tensor product $H \hat{\otimes} H$. Therefore, the statement follows directly from Lemma 3.2. \square

Using Lemma 3.4 instead of Theorem 2.4, we obtain the analog of Theorem 2.7 with the space of trace-class operators in place of the Hilbert space.

Proposition 3.5. *Let \mathcal{S}_1 be the trace-class operators on some Hilbert space, and let Y be a Banach space. Assigning to a projection $\pi: Y^{**} \rightarrow Y$ the projection $r_\pi: \mathcal{L}(\mathcal{S}_1, Y)^{**} \rightarrow \mathcal{L}(\mathcal{S}_1, Y)$, as in Theorem 2.6, established a natural bijection between the following classes.*

- (1) *contractive projections $Y^{**} \rightarrow Y$;*
- (2) *contractive projections $\mathcal{L}(\mathcal{S}_1, Y)^{**} \rightarrow \mathcal{L}(\mathcal{S}_1, Y)$.*

Restricted to projections with weak closed kernel, we obtain that the above correspondence gives a natural bijection between isometric preduals of Y and isometric preduals of $\mathcal{L}(\mathcal{S}_1, Y)$.*

Corollary 3.6. *Let \mathcal{S}_1 be the trace-class operators on some Hilbert space, let Y be a Banach space, and let $F \subseteq \mathcal{L}(\mathcal{S}_1, Y)^*$ be an isometric predual. Then for each $a \in \mathcal{L}(\mathcal{S}_1)$, the right action of a on $\mathcal{L}(\mathcal{S}_1, Y)$ is continuous for the weak* topology induced by F .*

Remark 3.7. Using Lemma 3.2 inductively, the statement of Proposition 3.5 holds for any projective tensor power $H \hat{\otimes} \dots \hat{\otimes} H$ of a Hilbert space in place of \mathcal{S}_1 .

Remark 3.8. The ‘standard’ predual of $\mathcal{S}_1(H)$ is $\mathcal{K}(H) \subseteq \mathcal{L}(H) = \mathcal{S}_1(H)^*$; see Paragraph 3.1. We obtain isometric isomorphisms

$$\mathcal{L}(\mathcal{S}_1(H)) \cong \mathcal{L}(\mathcal{S}_1(H), \mathcal{K}(H)^*) \cong (\mathcal{S}_1(H) \hat{\otimes} \mathcal{K}(H))^*.$$

We consider $\mathcal{S}_1(H) \hat{\otimes} \mathcal{K}(H)$ as the ‘standard’ predual of $\mathcal{L}(\mathcal{S}_1(H))$. Then right multiplication by an element from $\mathcal{L}(\mathcal{S}_1(H))$ is continuous for the induced weak* topology; see Corollary 3.6. However, this is not the case for left multiplication. Indeed, a Banach space X is reflexive if and only if left multiplication by elements from $\mathcal{L}(X)$ is weak* continuous (for any predual); see [GT16, Corollary 7.4].

Example 3.9. By Proposition 3.5, there is a natural correspondence between isometric preduals of $\mathcal{L}(\mathcal{S}_1(H))$ and isometric preduals of $\mathcal{S}_1(H)$. The compact operators on H form the canonical isometric predual of $\mathcal{S}_1(H)$.

However, if H is infinite-dimensional, then $\mathcal{S}_1(H)$ has also many other isometric preduals. Indeed, if H is infinite-dimensional, then the diagonal operators in $\mathcal{S}_1(H)$ form an isometric copy of $\ell^1 = \ell^1(\mathbb{N})$ that is closed for the ‘standard’ weak* topology induced by the compact operators. Since ℓ^1 does not have a strongly unique isometric predual, neither does $\mathcal{S}_1(H)$.

Theorem 3.10. *Let H be a Hilbert space. Then, the ‘standard’ predual $\mathcal{K}(H)$ of $\mathcal{S}_1(H)$ makes multiplication in $\mathcal{S}_1(H)$ separately weak* continuous. Moreover, $\mathcal{K}(H)$ is the only such predual: If $F \subseteq \mathcal{S}_1(H)^* = \mathcal{L}(H)$ is a (not necessarily isometric) predual that makes multiplication in $\mathcal{S}_1(H)$ separately weak* continuous, then $F = \mathcal{K}(H)$. In particular, every predual of $\mathcal{S}_1(H)$ making multiplication separately weak* continuous is automatically an isometric predual.*

Proof. Given $a \in \mathcal{L}(H)$, we let φ_a denote the functional on $\mathcal{S}_1(H)$ given by

$$\langle \varphi_a, x \rangle := \text{tr}(ax),$$

for $x \in \mathcal{S}_1(H)$. This identifies $\mathcal{S}_1(H)^*$ with $\mathcal{L}(H)$.

The multiplication on $\mathcal{S}_1(H)$ induces a $\mathcal{S}_1(H)$ -bimodule structure on its dual; see [GT16, Paragraph A.4]. Let us recall some details. Given $a \in \mathcal{S}_1(H)$, let $L_a, R_a: \mathcal{S}_1(H) \rightarrow \mathcal{S}_1(H)$ be left and right multiplication with a , respectively, that is, $L_a(x) := ax$ and $R_a(x) := xa$, for $x \in \mathcal{S}_1(H)$. Then the left action of a on $\mathcal{S}_1(H)^*$ is given by R_a^* , and the right action is given by L_a^* . Thus, given $a \in \mathcal{S}_1(H)$ and $b \in \mathcal{L}(H)$, we have

$$\langle \varphi_b a, x \rangle = \langle L_a^*(\varphi_b), x \rangle = \langle \varphi_b, ax \rangle = \text{tr}(bax) = \langle \varphi_{ba}, x \rangle,$$

for $x \in \mathcal{S}_1(H)$, and therefore $\varphi_b a = \varphi_{ba}$. Similarly, we have

$$\langle a \varphi_b, x \rangle = \langle R_a^*(\varphi_b), x \rangle = \langle \varphi_b, xa \rangle = \text{tr}(bxa) = \text{tr}(abx) = \langle \varphi_{ab}, x \rangle,$$

for $x \in \mathcal{S}_1(H)$, and therefore $a \varphi_b = \varphi_{ab}$.

Let $F \subseteq \mathcal{S}_1(H)^* = \mathcal{L}(H)$ be a predual. Then left (right) multiplication on $\mathcal{S}_1(H)$ is $\sigma(\mathcal{S}_1(H), F)$ -continuous if and only if F is a right (left) $\mathcal{S}_1(H)$ -submodule of $\mathcal{L}(H)$; see [GT16, Proposition B.8]. Given $a \in \mathcal{K}(H)$ and $b \in \mathcal{S}_1(H)$, we have $ab, ba \in \mathcal{K}(H)$. Thus, the predual $\mathcal{K}(H)$ is invariant under the left and right action by $\mathcal{S}_1(H)$, which implies that it makes multiplication in $\mathcal{S}_1(H)$ separately weak* continuous.

Conversely, let $F \subseteq \mathcal{S}_1(H)^* = \mathcal{L}(H)$ be a predual making multiplication in $\mathcal{S}_1(H)$ separately weak* continuous. Then F is invariant under the left and right action of $\mathcal{S}_1(H)$ on $\mathcal{L}(H)$. We have shown above that the left (right) action of $a \in \mathcal{S}_1(H)$ on $\mathcal{L}(H)$ is simply given by right (left) multiplication with a .

Claim: The set $F \cap \mathcal{K}(H)$ is a closed, two-sided ideal in $\mathcal{K}(H)$. To verify the claim, let $a \in F \cap \mathcal{K}(H)$, and let $b \in \mathcal{K}(H)$. Given a finite-dimensional subspace $D \subseteq H$, let p_D be the orthogonal projection onto D . We order the finite-dimensional subspaces of H by inclusion. Since $b \in \mathcal{K}(H)$, we have $\lim_D \|p_D b - b\| = 0$ and therefore

$$\lim_D \|ap_D b - ab\| = 0.$$

For each D , we have $p_D b \in \mathcal{S}_1(H)$. Since F is invariant under right multiplication by $\mathcal{S}_1(H)$, it follows that $ap_D b \in F \cap \mathcal{K}(H)$. Since F is norm-closed, we deduce that $ab \in F \cap \mathcal{K}(H)$. Analogously, one shows that $F \cap \mathcal{K}(H)$ is a left ideal in $\mathcal{K}(H)$, which proves the claim.

The only closed, two-sided ideals of $\mathcal{K}(H)$ are $\{0\}$ and $\mathcal{K}(H)$. It is easy to see that $F \cap \mathcal{K}(H) \neq \{0\}$. Thus, $\mathcal{K}(H) \subseteq F$. Since both $\mathcal{K}(H)$ and F are preduals of $\mathcal{S}_1(H)$, it follows that $\mathcal{K}(H) = F$, as desired. \square

Corollary 3.11. *Let $\mathcal{S}_1(H)$ be the trace-class operators on a Hilbert space H . Then every Banach algebra isomorphism $\mathcal{S}_1(H) \rightarrow \mathcal{S}_1(H)$ is weak* continuous (for the ‘standard’ predual $\mathcal{K}(H)$.)*

Remark 3.12. A *dual Banach algebra* is a Banach algebra A together with a predual $F \subseteq A^*$ making the multiplication in A separately weak* continuous. This concept was introduced by Runde, [Run02, Definition 4.4.1, p.108], and extensively studied by Daws, [Daw04], [Daw07], [Daw11]. Theorem 3.10 states that the trace-class operators with their ‘standard’ predual of compact operators form a dual Banach algebra. Moreover, the compact operators are the only predual making the trace-class operators into a dual Banach algebra.

REFERENCES

- [Bla06] B. BLACKADAR, *Operator algebras, Encyclopaedia of Mathematical Sciences* **122**, Springer-Verlag, Berlin, 2006, Theory of C^* -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR 2188261 (2006k:46082). Zbl 1092.46003.
- [Chu81] C.-H. CHU, A note on scattered C^* -algebras and the Radon-Nikodým property, *J. London Math. Soc. (2)* **24** (1981), 533–536. MR 635884. Zbl 0438.46041.
- [Daw04] M. DAWS, Banach algebras of operators, Ph.D Thesis, The University of Leeds, School of Mathematics, Department of Pure Mathematics, 2004.
- [Daw07] M. DAWS, Dual Banach algebras: representations and injectivity, *Studia Math.* **178** (2007), 231–275. MR 2289356. Zbl 1115.46038.
- [Daw11] M. DAWS, A bicommutant theorem for dual Banach algebras, *Math. Proc. R. Ir. Acad.* **111A** (2011), 21–28. MR 2851064. Zbl 1285.46036.
- [GT16] E. GARDELLA and H. THIEL, Preduals and complementation of spaces of bounded linear operators, preprint (arXiv:1609.05326 [math.FA]), 2016.
- [God89] G. GODEFROY, Existence and uniqueness of isometric preduals: a survey, in *Banach space theory (Iowa City, IA, 1987)*, *Contemp. Math.* **85**, Amer. Math. Soc., Providence, RI, 1989, pp. 131–193. MR 983385. Zbl 0674.46010.
- [GS88] G. GODEFROY and P. D. SAPHAR, Duality in spaces of operators and smooth norms on Banach spaces, *Illinois J. Math.* **32** (1988), 672–695. MR 955384 (89j:47026). Zbl 0631.46015.
- [Mar10] L. W. MARCOUX, Projections, commutators and Lie ideals in C^* -algebras, *Math. Proc. R. Ir. Acad.* **110A** (2010), 31–55. MR 2666670. Zbl 1279.46041.
- [PT67] C. PEARCY and D. TOPPING, Sums of small numbers of idempotents, *Michigan Math. J.* **14** (1967), 453–465. MR 0218922. Zbl 0156.38102.
- [Run02] V. RUNDE, *Lectures on amenability, Lecture Notes in Mathematics* **1774**, Springer-Verlag, Berlin, 2002. MR 1874893 (2003h:46001). Zbl 0999.46022.

HANNES THIEL MATHEMATISCHES INSTITUT, FACHBEREICH MATHEMATIK UND INFORMATIK DER UNIVERSITÄT MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY.

E-mail address: hannes.thiel@uni-muenster.de

URL: www.math.uni-muenster.de/u/hannes.thiel/